Zadatak 1

Za funkciju $f(x)=\log(x+1)$ poznate su vrijednosti $f(1.5)$, $f(2)$ i $f(2.5)$. Odredite $f'(2)$:

a) Hermiteovom metodom ako je još poznato i $f'(2.5)$, (20)

b) koristeći kubni splajn ako su poznate vrijednosti $f'(1.5)$ i $f'(2.5)$, (10)

c) numeričkim diferenciranjem. (10)

Izračunajte pravu grešku u sva tri slučaja.

Rješenje. a) $f'(x)=\frac{1}{\ln x}\cdot\frac{1}{x+1}$

x_i	y_i	$f^{[1]}$	$f^{[2]}$	$f^{[3]}$
$x_{-1}=2$	$y_{-1}=0.47712$			
		$f'(x_{-1})=?$		
$x_{-1}=2$	$y_{-1}=0.47712$		$f[x_{-1},x_{-1},x_0]=?$	
		$f[x_{-1},x_0]=0.15836$		$f[x_{-1},x_{-1},x_0,x_1]=?$
$x_0=1.5$	$y_0=0.39794$		$f[x_{-1},x_0,x_1]=-0.02446$	
		$f[x_0,x_1]=0.14613$		$f[x_{-1},x_0,x_1,x_1]=0.00482$
$x_1=2.5$	$y_1=0.54407$		$f[x_0,x_1,x_1]=-0.02205$	
		$f'(x_1)=0.12408$		
$x_1=2.5$	$y_1=0.54407$			

$$\frac{-0.02446-f[x_{-1},x_{-1},x_0]}{0.5}=0.00482\Rightarrow f[x_{-1},x_{-1},x_0]=-0.02687$$

$$\Rightarrow\frac{0.15836-f'(x_{-1})}{-0,5}=-0.02687\Rightarrow f'(2)=0.14493.$$

Kako je prava vrijednost $f'(2)=0.14476$ za pravu grešku imamo $|0.14493-0.14476|=0.00017$.

b)

x_i	y_i	$f[x_i,x_{i+1}]$
$x_0=1.5$	$y_0=0.39794$	
		$f[x_0,x_1]=0.15836$
$x_1=2$	$y_1=0.47712$	
		$f[x_1,x_2]=0.1339$
$x_2=2.5$	$y_2=0.54407$	

$$\Rightarrow 0.5s_0+2s_1+0.5s_2=0.43839.$$

Rješenje je $s_1=0.14475$, a prava grška $|0.14475-0.14476|=0.00001$.

c)

$$f'(2)=\frac{1}{2\cdot 0.5}\left(f(2.5)-f(1.5)\right)=0.14613.$$

Prava greška: $|0.14613-0.14476|=0.14613$.

Zadatak 2 *Simpsonovom metodom s točnošću većom od* 0.1 *izračunajte* $\int_0^1\frac{dx}{4x+1}$. *Odredite pravu grešku.* (15)

Rješenje.

$$f(x)=\frac{1}{4x+1}\Rightarrow f'(x)=-\frac{4}{(4x+1)^2}\Rightarrow f''(x)=\frac{32}{(4x+1)^3}\Rightarrow f'''(x)=-\frac{384}{(4x+1)^4}\Rightarrow f^{iv}(x)=\frac{6144}{(4x+1)^5}$$

$$\Rightarrow M_4=f(0)=6144\Rightarrow\frac{1}{180}\cdot 6144h^4<0.1\Rightarrow 2n>4.29827\Rightarrow 2n=6.$$

x_i	$f(x_i)$
$x_0=0$	$f(x_0)=1$
$x_1=1/6$	$f(x_1)=0.6$
$x_2=1/3$	$f(x_2)=0.42857$
$x_3=1/2$	$f(x_3)=0.33333$
$x_4=2/3$	$f(x_4)=0.27273$
$x_5=5/6$	$f(x_3)=0.23077$
$x_6=1$	$f(x_4)=0.2$

$$\Rightarrow I_6=\frac{1}{18}(1+4(0.6+0.33333+0.23077)+2(0.42857+0.27273)+0.2)=0.40238.$$

Kako je $\int_0^1 \frac{dx}{4x+1} = \frac{1}{4}\ln(4x+1)\Big|_0^1 = 0.40236$, prava greška je $|0.40236 - 0.40238| = 0.9 \cdot 10^{-3}$.

Zadatak 3 *Koristeći Laplaceovu transformaciju odredite rješenje diferencijalne jednadžbe $x''(t) + x(t) = 2e^{-t}$ uz početne uvjete $x(0) = 2, x'(0) = -1$.* *(15)*

Rješenje.
$$\mathcal{L}(x'') = p^2X - px_0 - x_0' = p^2X - 2p + 1 \Rightarrow p^2X - 2p + 1 + X = \frac{2}{p+1}$$
$$\Rightarrow X = \frac{2p^2+p+1}{(p+1)(p^2+1)} = \frac{1}{p+1} + \frac{p}{p^2+1}$$
$$\Rightarrow x(t) = e^{-t} + \cos t.$$

Zadatak 4 *Diferencijalnu jednadžbu $y' = \frac{x}{y^3}$, $y(0) = 1$ na intervalu $[0,1]$ s korakom $h = 0.5$ približno riješite Eulerovom metodom, te Runge-Kutta metodom i ocjenite koja je metoda točnija u točki $x = 1$ (izračunajte pravu grešku).* *(15)*

Rješenje. Pravo rješenje:
$$y^3dy = xdx \Rightarrow \frac{y^4}{4} = \frac{x^2}{2} + C \Rightarrow C = \frac{1}{4} \Rightarrow y^4 = 2x^2 + 1 \Rightarrow y(1) = 1.31607.$$

Eulerova metoda:
$$y_1 = 1 + 0.5 \cdot \frac{0}{1} = 1$$
$$y_2 = 1 + 0.5 \cdot \frac{0.5}{1} = 1.25$$

Prava greška: $|1.25 - 1.31607| = 0.06607$.

Runge-Kuttina metoda:
$$K_1^0 = 0, \quad K_2^0 = 0.125, \quad K_3^0 = 0.10421, \quad K_4^0 = 0.18569$$
$$\triangle y_0 = 0.10735 \Rightarrow y_1 = 1.10735$$
$$K_1^1 = 0.18411, \quad K_2^1 = 0.21734, \quad K_3^1 = 0.20855, \quad K_4^1 = 0.21943$$
$$\triangle y_1 = 0.20922 \Rightarrow y_2 = 1.31657$$

Prava greška: $|1.31657 - 1.31607| = 0.0005$.

Točnija je Runge-Kuttina metoda.

Zadatak 5 *Metodom zlatnog reza s greškom manjom od $\varepsilon = 0.5$ odredite minimum funkcije $f(x) = (x+1)^2$ na intervalu $[-1.5, 0]$.* *(15)*

Rješenje. Kako je $a^{(0)} = -1.5$ i $c^{(0)} = 0$ imamo
$$\frac{b^{(0)} + 1.5}{1.5} = \frac{3-\sqrt{5}}{2} \Rightarrow b^{(0)} = -0.92705.$$

Kako je još
$$f(a^{(0)}) = f(-1.5) = 0.25, \quad f(c^{(0)}) = f(0) = 1, \quad f(b^{(0)}) = 0.00532$$
$$\Rightarrow f(b^{(0)}) < f(a^{(0)}) \;\text{ i } f(b^{(0)}) < f(c^{(0)}),$$

početne su točke dobro odabrane.

Sada,
$$x^{(0)} = c^{(0)} + a^{(0)} - b^{(0)} = -0.57295, \quad f(x^{(0)}) = 0.18237 > f(b^{(0)})$$
$$\Rightarrow a^{(1)} = -1.5, \;\; b^{(1)} = a^{(0)} + x^{(0)} - b^{(0)} = -1.1459, \;\; c^{(1)} = -0.57295, \;\; |c^{(1)} - a^{(1)}| = 0.92705 > 0.5$$
$$\Rightarrow x^{(1)} = c^{(1)} + a^{(1)} - b^{(1)} = -0.92705, \quad f(x^{(1)}) = 0.00532 < f(b^{(1)}) = 0.02129$$
$$\Rightarrow a^{(2)} = -1.1459, \quad b^{(2)} = -0.92705, \quad c^{(2)} = -0.57295, \quad |c^{(2)} - a^{(2)}| = 0.57295 > 0.5$$
$$\Rightarrow x^{(2)} = c^{(2)} + a^{(2)} - b^{(2)} = -0.7918, \quad f(x^{(2)}) = 0.04335 > f(b^{(2)})$$
$$\Rightarrow a^{(3)} = -1.1459, \;\; b^{(3)} = a^{(2)} + x^{(2)} - b^{(2)} = -1.01065, \;\; c^{(3)} = -0.7918, \;\; |c^{(3)} - a^{(3)}| = 0.3541 < 0.5$$
$$\Rightarrow x^* = (a^{(3)} + c^{(3)})/2 = -0.96885.$$