Zadatak 1 *S točnošću većom od* 10^{-5} *odredite* $\ln 54$*. Izračunajte ukupnu grešku.* *(10)*

Rješenje.

$$54 = 2^6 \cdot \frac{54}{64} \Rightarrow \frac{27}{32} = \frac{1-x}{1+x} \Rightarrow x = \frac{5}{59}$$

$$n = 2 \Rightarrow R_5\left(\frac{5}{59}\right) \le \frac{9}{4}\frac{\left(\frac{5}{59}\right)^5}{5} = 0.1966 \cdot 10^{-5} < 0.25 \cdot 10^{-5}$$

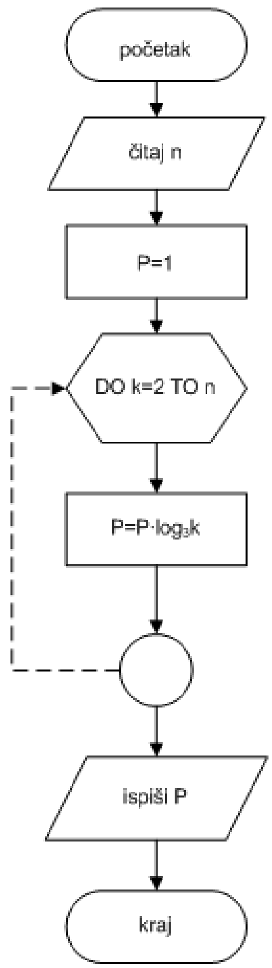
$$\Rightarrow \ln\frac{27}{32} = -2\left[\frac{5}{59} + \frac{1}{3}\left(\frac{5}{59}\right)^3\right] = -2[0.084746 + 0.000203] = -0.169898$$

$$\ln 54 = 6\ln 2 + \ln\frac{27}{32} = 4.158883 - 0.169898 = 3.988985$$

$$\varepsilon = 0.1966 \cdot 10^{-5} + 2 \cdot 0.5 \cdot 10^{-6} + 0 = 0.2966 \cdot 10^{-5} < 10^{-5}.$$

Zadatak 2 *Opišite dijagram toka i napišite program u Mathematica-i za algoritam koji za zadani cijeli broj* $n \ge 2$ *(ulazna informacija) računa* $\log_3 2 \cdot \log_3 3 \cdot \ldots \cdot \log_3 n$*.* *(15)*

Rješenje.




```
n = 10;
P = 1;
For[k = 2, k <= n, k = k + 1, P = P * Log[3, k]];
Print[N[P]]
```

```
26.7325
```

Slika 1:

Zadatak 3 *Jacobijevom metodom (jednom iteracijom) odredite približno rješenje sustava*

$$\begin{aligned} 6x_1 + 2x_2 &= 4 \\ x_1 + 2x_2 &= 3. \end{aligned}$$

Odredite pravu grešku. *(15)*

Rješenje.

$$D = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow x^{(0)} = D^{-1}b = \frac{1}{12} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{3}{2} \end{bmatrix}$$

$$\Rightarrow x^{(1)} = \frac{1}{12} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{3}{2} \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{6} \\ \frac{7}{6} \end{bmatrix}$$

Pravo rješenje:

$$\begin{aligned} 6x_1 + 2x_2 &= 4 \\ x_1 + 2x_2 &= 3 \end{aligned} \Leftrightarrow \begin{aligned} 6x_1 + 2x_2 &= 4 \\ -6x_1 - 12x_2 &= -18 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{7}{5} \end{bmatrix}$$

Prava greška:

$$\varepsilon = \sqrt{(0.2 - 0.16667)^2 + (1.4 - 1.16667)^2} = 0.23569$$

Zadatak 4 *Odredite vezu oblika* $\frac{a}{x^2} + \frac{b}{y} = 2$ *ako je*

x_k	-3	-2	-1
y_k	1.1	1.2	2

(15)

Rješenje.

$$\frac{b}{y} = 2 - \frac{a}{x^2} \Rightarrow \frac{1}{y} = \frac{2}{b} - \frac{a}{b} \cdot \frac{1}{x^2} \Rightarrow \bar{y} = a_0 + a_1\bar{x}, \quad \bar{y} = \frac{1}{y}, \bar{x} = \frac{1}{x^2}, a_0 = \frac{2}{b}, a_1 = -\frac{a}{b}$$

$\bar{x}_i$	0.11	0.25	1
$\bar{y}_i$	0.91	0.83	0.5

$$\Rightarrow \sum_{i=0}^{2} \bar{x}_i = 1.36, \sum_{i=0}^{2} \bar{x}_i^2 = 1.075, \sum_{i=0}^{2} \bar{x}_i\bar{y}_i = 0.808, \sum_{i=0}^{2} \bar{y}_i = 2.24 \Rightarrow a_0 = 0.95, a_1 = -0.45$$

$$\Rightarrow \ b = \frac{2}{a_0} = 2.1, a = -ba_1 = 0.95 \Rightarrow \frac{0.95}{x^2} + \frac{2.1}{y} = 2.$$

Zadatak 5 *Odredite polinom prvog stupnja koji u smislu metode najmanjih kvadrata najbolje aproksimira funkciju* $f(x) = |\sin \frac{x}{2}|$ *na intervalu* $[-\pi, \pi]$. *(15)*

Rješenje.

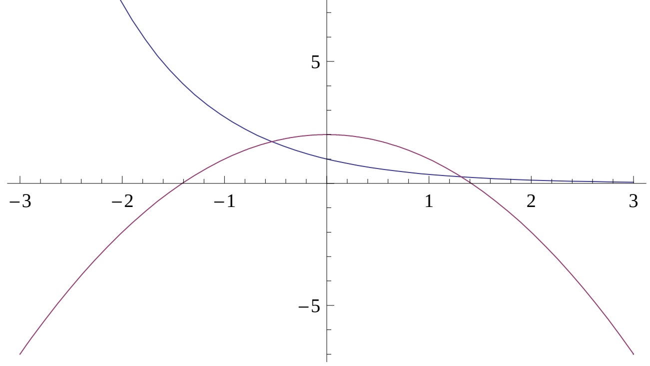
$$\int_{-\pi}^{\pi} x dx = 0, \ \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3}, \ \int_{-\pi}^{\pi} \left|\sin \frac{x}{2}\right| dx = -\int_{-\pi}^{0} \sin \frac{x}{2} dx + \int_{0}^{\pi} \sin \frac{x}{2} dx = 4,$$

$$\int_{-\pi}^{\pi} x \left|\sin \frac{x}{2}\right| dx = -\int_{-\pi}^{0} x \sin \frac{x}{2} dx + \int_{0}^{\pi} x \sin \frac{x}{2} dx = 0$$

$$a_0 = \frac{2}{\pi}, \ a_1 = 0 \Rightarrow \varphi(x) = \frac{2}{\pi}$$

Zadatak 6 *Pripremite za Newtonovu metodu i izračnajte prvu aproksimaciju negativne nultočke jednadžbe* $e^{-x} + x^2 = 2$. *(15)*

Rješenje.



Slika 2:

$$f(x) = e^{-x} + x^2 - 2, \quad f(-1) = 1.72 > 0, \quad f(0) = -2 < 0 \Rightarrow \text{ nultočka je unutar intervala } [-1, 0]$$

$$f'(x) = -e^{-x} + 2x < 0 \Rightarrow m_1 = |f'(0)| = 1 \Rightarrow f''(x) = e^{-x} + 2 > 0 \Rightarrow M_2 = f''(-1) = e + 2$$

$$\Rightarrow \frac{e+2}{2}(x_n - x_{n-1})^2 < \varepsilon$$

$$x_0 = -1 \Rightarrow x_1 = -1 - \frac{f(-1)}{f'(-1)} = -0.63582.$$

Zadatak 7 *Metodom iteracije s točnošću većom od* 10^{-2} *odredite približno rješenje sustava* $x^2 = y + 3\log y, \quad 2y^2 = xy + 5y - 1$, *uzimajući za početne vrijednosti* $x_0 = 2.2, \quad y_0 = 3.4$. *(15)*

Rješenje.

$$x = \sqrt{y + 3\log y} = \phi_1(x,y), \; y = \sqrt{\frac{1}{2}(xy + 5y - 1)} = \phi_2(x,y)$$

$$x_1 = \phi_1(x_0, y_0) = 2.235, \; y_1 = \phi_2(x_0, y_0) = 3.426,$$

$$x_2 = \phi_1(x_1, y_1) = 2.243, \; y_2 = \phi_2(x_1, y_1) = 3.449,$$

$$x_3 = \phi_1(x_2, y_2) = 2.254, \; y_3 = \phi_2(x_2, y_2) = 3.463,$$

$$x_4 = \phi_1(x_3, y_3) = 2.254, \; y_4 = \phi_2(x_3, y_3) = 3.473,$$

$$x_5 = \phi_1(x_4, y_4) = 2.257, \; y_3 = \phi_2(x_2, y_2) = 3.478.$$

Zadatak 8 *Za funkciju* $f(x) = \arccos\frac{x}{2}$ *poznate su vrijednosti* $f(0)$ *i* $f(1)$. *Odredite* $f'(-1)$*:*

a) Hermiteovom metodom ako je još poznato i $f'(1)$, *(15)*

b) koristeći kubni splajn ako su poznate vrijednosti $f(-1), f''(-1)$ *i* $f''(1)$, *(15)*

c) numeričkim diferenciranjem ako je poznata i vrijednost $f(-1)$. *(10)*

Izračunajte pravu grešku u sva tri slučaja.

Rješenje. a) $f'(x) = -\frac{1}{2}\frac{1}{\sqrt{1-\frac{x^2}{4}}}$

x_i	y_i	$f^{[1]}$	$f^{[2]}$
$x_{-1} = -1$	$y_{-1} = ?$		
		$f'(x_{-1}) = ?$	
$x_{-1} = -1$	$y_{-1} = ?$		$f[x_{-1}, x_{-1}, x_0] = ?$
		$f[x_{-1}, x_0] = ?$	
$x_0 = 0$	$y_0 = \frac{\pi}{2}$		$f[x_{-1}, x_0, x_1] = ?$
		$f[x_0, x_1] = -\frac{\pi}{6}$	
$x_1 = 1$	$y_1 = \frac{\pi}{3}$		$f[x_0, x_1, x_1] = -0.06328$
		$f'\left(\frac{\pi}{3}\right) = -0.58688$	
$x_1 = 1$	$y_1 = \frac{\pi}{3}$		

$$\frac{-\frac{\pi}{6} - f[x_{-1}, x_0]}{2} = -0.06328 \Rightarrow f[x_{-1}, x_0] = -0.39704$$

$$\Rightarrow \frac{-0.39704 - f'(x_{-1})}{1} = -0.06328 \Rightarrow f'(-1) = -0.33376.$$

Kako je prava vrijednost $f'(-1) = -0.57735$ za pravu grešku imamo $|-0.57735+0.33376| = 0.24359$.

b)

x_i	y_i	$f[x_i, x_{i+1}]$
$x_0 = -1$	$y_0 = \frac{2\pi}{3}$	
		$f[x_0, x_1] = -\frac{\pi}{6}$
$x_1 = 0$	$y_1 = \frac{\pi}{2}$	
		$f[x_1, x_2] = -\frac{\pi}{6}$
$x_2 = 1$	$y_2 = \frac{\pi}{3}$	

$$\Rightarrow s_0 + 4s_1 + s_2 = 3\left(-\frac{\pi}{6} - \frac{\pi}{6}\right) = -\pi.$$

Kako je i $f''(x) = -\frac{x}{8\left(1-\frac{x^2}{4}\right)^{3/2}}$ imamo

$$2s_0 + s_1 = -\frac{\pi}{2} - \frac{1}{2} \cdot 0.19245 = -1.66702,$$

$$s_1 + 2s_2 = -\frac{\pi}{2} - \frac{1}{2} \cdot 0.19245 = -1.66702.$$

Rješenje sustava je $s_0 = -0.58775$, a prava grška $|-0.57735 + 0.58775| = 0.0104$.

c)

$$f'(-1) = \frac{1}{2}(-3f(-1) + 4f(0) - f(1)) = -0.5236.$$

Greška: $|-0.57735 + 0.5236| = 0.05375$.

Zadatak 9 *Simpsonovom metodom s točnošću većom od* 10^{-6} *izračunajte* $\int_3^4 \ln(x+2)dx$. *Odredite pravu grešku.*

(15)

Rješenje.

$$f(x) = \ln(x+2) \Rightarrow f'(x) = \frac{1}{x+2} \Rightarrow f''(x) = -\frac{1}{(x+2)^2} \Rightarrow f'''(x) = \frac{2}{(x+2)^3} \Rightarrow f^{iv}(x) = -\frac{6}{(x+2)^4}$$

$$\Rightarrow M_4 = -f(3) = 0.0096 \Rightarrow \frac{h^4}{180} \cdot 0.0096 < 10^{-6} \Rightarrow 2n > 2.7 \Rightarrow 2n = 4.$$

x_i	$f(x_i)$
$x_0 = 3$	$f(x_0) = 1.6094379$
$x_1 = 3.25$	$f(x_1) = 1.6582281$
$x_2 = 3.5$	$f(x_2) = 1.7047481$
$x_3 = 3.75$	$f(x_3) = 1.7491999$
$x_4 = 4$	$f(x_4) = 1.7917595$

$$\Rightarrow I_4 = \frac{1}{12}(1.6094379+4(1.6582281+1.7491999)+2\cdot 1.7047481+1.7917595) = 1.7033671.$$

Kako je $\int_3^4 \ln(x+2)dx = x\ln(x+2)\Big|_3^4 - \int_3^4 \frac{x}{x+2}dx = 1.70336725$, prava greška je $|1.70336725 - 1.7033671| = 0.15 \cdot 10^{-6}$.

Zadatak 10 *Koristeći Laplaceovu transformaciju odredite rješenje diferencijalne jednadžbe* $x''(t) + x'(t) = 2e^{-2t}$ *uz početne uvjete* $x(0) = 5, x'(0) = -2$. *(15)*

Rješenje.

$$\mathcal{L}(x') = pX - x_0 = pX - 5,\ \mathcal{L}(x'') = p^2X - px_0 - x_0' = p^2X - 5p + 2 \Rightarrow p^2X - 5p + 2 + pX - 5 = \frac{2}{p+2}$$

$$\Rightarrow X = \frac{5p^2 + 13p + 8}{p(p+1)(p+2)} = \frac{4}{p} + \frac{1}{p+2}$$

$$\Rightarrow x(t) = 4 + e^{-2t}.$$

Zadatak 11 *Diferencijalnu jednadžbu* $y' = x^3y,\ y(0) = 1$ *na intervalu* $[0,1]$ *s korakom* $h = 0.5$ *približno riješite Eulerovom metodom, te Picardovom metodom u dvije iteracije i ocjenite koja je metoda točnija u točki* $x = 0.5$ *(izračunajte pravu grešku).* *(15)*

Rješenje. Pravo rješenje:

$$\frac{dy}{y} = x^3 dx \Rightarrow \ln y = \frac{x^4}{4} + \ln C \Rightarrow C = 1 \Rightarrow y = e^{\frac{x^4}{4}} \Rightarrow y(0.5) = 1.01575.$$

Eulerova metoda:

$$y_1 = 1 + 0.5 \cdot 0 = 1$$

$$y_2 = 1 + 0.5 \cdot 0.5^3 = 1.0625$$

Prava greška: $|1.01575 - 1.0625| = 0.04675$.

Picardova metoda:

$$y_1(x) = 1 + \int_0^x x^3 dx = 1 + \frac{x^4}{4} \Rightarrow y_2(x) = 1 + \int_0^x \left(x^3 + \frac{x^7}{4}\right) dx = 1 + \frac{x^4}{4} + \frac{x^8}{32} \Rightarrow y_2(0.5) = 1.01758.$$

Prava greška: $|1.01575 - 1.01758| = 0.183 \cdot 10^{-2}$.

Točnija je Picardova metoda.

Zadatak 12 *Koristeći shemu konačnih razlika približno rješite rubni problem za parcijalnu diferencijalnu jednadžbu drugog reda:*

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x - y, & na \quad S = [0,1] \times [-1,0] \\ u(x,y) = x, & na \quad \Gamma = \partial S \end{cases}$$

$s\ h = k = 0.5.$ *(15)*

Rješenje.

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1j} - 2u_{ij} + u_{i,j+1}}{h^2} = x_i - y_j \Rightarrow u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1} = 0.5^2(x_i - y_j).$$

Kako je

$$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1, \quad y_0 = -1,\ y_1 = -0.5,\ y_2 = 0,$$

zbog rubnog uvjeta imamo

$$u_{00} = u_{01} = u_{02} = 0,\ u_{10} = u_{12} = 0.5,\ u_{20} = u_{21} = u_{22} = 1.$$

Sada, za $i = j = 1$ imamo

$$u_{10} + u_{01} - 4u_{11} + u_{21} + u_{12} = 0.5^2 \Rightarrow u_{11} = 0.04375.$$