NUMERIČKE METODE I PROGRAMIRANJE
(formule za prvi parcijalni ispit)

Izračunavanje vrijednosti nekih elementarnih funkcija

1. $f(x) = e^x$

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}, \quad |R_{n+1}(x)| \leq \frac{3x^{n+1}}{(n+1)!}, \quad 0 \leq x \leq 1$$

2. $f(x) = \sin x$

$$T_{2n-1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}, \quad |R_{2n+1}(x)| \leq \frac{x^{2n+1}}{(2n+1)!}, \quad 0 \leq x \leq \frac{\pi}{4}$$

3. $f(x) = \cos x$

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!}, \quad |R_{2n+2}(x)| \leq \frac{x^{2n+2}}{(2n+2)!}, \quad 0 \leq x \leq \frac{\pi}{4}$$

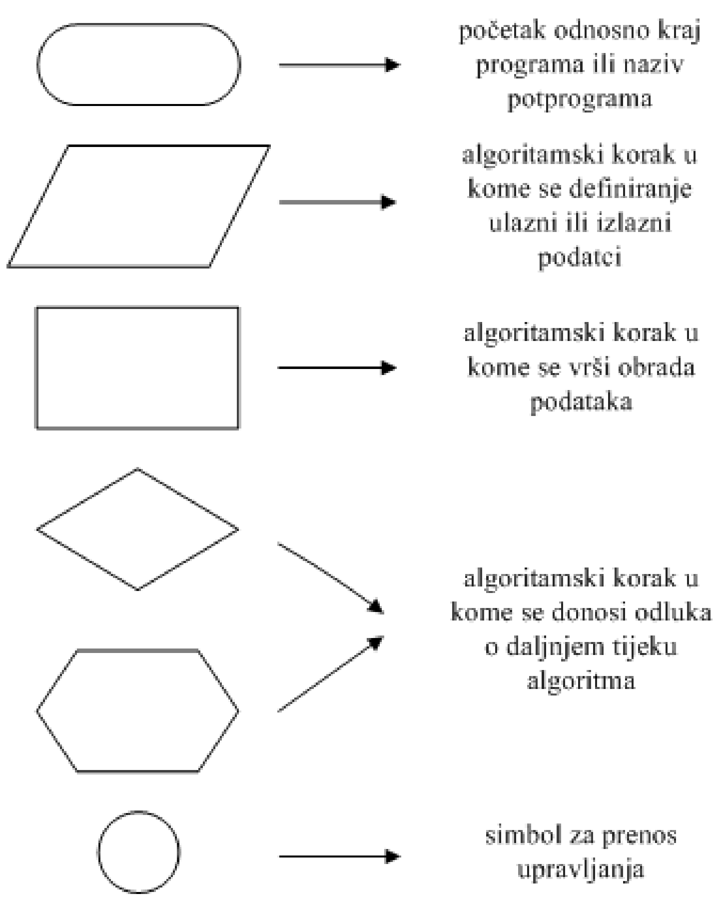
4. $f(x) = \ln(1+x), \quad |x| \leq 1,$

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n}, \quad |R_{n+1}(x)| \leq \frac{x^{n+1}}{n+1}, \quad 0 \leq x \leq 1$$

Za $f(x) = \ln\frac{1-x}{1+x} = \ln(1-x) - \ln(1+x)$

$$T_{2n-1}(x) = -2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1}\right), \quad |R_{2n+1}(x)| \leq \frac{9}{4}\cdot\frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

Algoritmi, dijagrami toka, pseudoprogrami



Slika 1:

Programiranje pomoću programskog paketa *Mathematica*

IF struktura: If[uvjetni izraz, grupa naredbi (1)(true), grupa naredbi (2) (false)]

DO struktura: Do[tijelo petlje,{k, kmin, kmax, korak}]

FOR struktura: For[početak, uvjetni izraz, korak, tijelo petlje]

Numeričke metode u linearnoj algebri

Jacobijeva metoda

$$D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & 0 & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & 0 \end{bmatrix},$$

$$x^{(k+1)} = D^{-1}(b + Nx^{(k)}) = D^{-1}Nx^{(k)} + D^{-1}b$$

Gauss-Seidelova metoda

$$L = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad U = -\begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$x^{(k+1)} = L^{-1}(b + Ux^{(k)}), \quad k = 0, 1, \ldots$$

ili

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \quad i = 1, \ldots, n.$$

Metoda najmanjih kvadrata

Diskretni slučaj

$$\varphi(x) = a_0 + a_1 x,$$

$$\begin{aligned} a_0 &= \frac{\sum_{k=0}^n x_k^2 \sum_{k=0}^n f(x_k) - \sum_{k=0}^n x_k \sum_{k=0}^n f(x_k) x_k}{(n+1)\sum_{k=0}^n x_k^2 - \left(\sum_{k=0}^n x_k\right)^2}, \\ a_1 &= \frac{(n+1)\sum_{k=0}^n x_k f(x_k) - \sum_{k=0}^n x_k \sum_{k=0}^n f(x_k)}{(n+1)\sum_{k=0}^n x_k^2 - \left(\sum_{k=0}^n x_k\right)^2}. \end{aligned}$$

Neprekidni slučaj

$$\varphi(x) = a_0 + a_1 x,$$

$$\begin{aligned} a_0 &= \frac{\int_a^b x^2 dx \int_a^b f(x) dx - \int_a^b x dx \int_a^b x f(x) dx}{(b-a)\int_a^b x^2 dx - \left(\int_a^b x dx\right)^2}, \\ a_1 &= \frac{(b-a)\int_a^b x f(x) dx - \int_a^b x dx \int_a^b f(x) dx}{(b-a)\int_a^b x^2 dx - \left(\int_a^b x dx\right)^2}. \end{aligned}$$

Fourierov polinom

$$\begin{aligned} x\in[0,2L],\quad T_m(x) &= A_0+A_1\cos\frac{\pi x}{L}+B_1\sin\frac{\pi x}{L}+A_2\cos\frac{2\pi x}{L}+B_2\sin\frac{2\pi x}{L}+\dots \\ &+ A_m\cos\frac{m\pi x}{L}+B_m\sin\frac{m\pi x}{L}. \end{aligned}$$

$$A_0=\frac{1}{2L}\int_0^{2L} f(x)dx,\quad A_k=\frac{1}{L}\int_0^{2L} f(x)\cos\frac{k\pi x}{L}dx,\quad B_k=\frac{1}{L}\int_0^{2L} f(x)\sin\frac{k\pi x}{L}dx,\quad k=1,\dots,m.$$

Greška:

$$\varepsilon_m=\int_0^{2L} f(x)^2dx-2L\left[A_0^2+\frac{1}{2}\sum_{i=1}^{m}(A_i^2+B_i^2)\right]$$

Nelinearne jednadžbe

Newtonova metoda (metoda tangente)

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\quad n=0,1,2,\dots$$

Ako je $f'(x)>0,\ f''(x)>0$ ili $f'(x)<0,\ f''(x)<0$ onda je $x_0=b$, a ako je $f'(x)>0,\ f''(x)<0$ ili $f'(x)<0,\ f''(x)>0$ onda je $x_0=a$.

Test za zadanu grešku ε:

$$|x_n-x_{n-1}|\le\sqrt{\frac{2m_1\varepsilon}{M_2}},$$

gdje je $m_1=\min_{x\in[a,b]}|f'(x)|,\quad M_2=\max_{x\in[a,b]}|f''(x)|$.

Metoda sekante

1. $f(a)>0\Rightarrow x_0=b$ i

$$x_{n+1}=x_n-\frac{f(x_n)}{f(x_n)-f(a)}(x_n-a),\qquad n=0,1,2,\dots$$

2. $f(a)<0\Rightarrow x_0=a$ i

$$x_{n+1}=x_n-\frac{f(x_n)}{f(b)-f(x_n)}(b-x_n),\qquad n=0,1,2,\dots$$

Test za zadanu grešku ε:

$$|x_n-x_{n-1}|\le\frac{m\cdot\varepsilon}{M-m},$$

gdje je $m\le|f'(x)|\le M, x\in[a,b]$.

Ili,

$$x_{n+1}=\frac{x_{n-1}f(x_n)-x_nf(x_{n-1})}{f(x_n)-f(x_{n-1})}$$

Test za zadanu grešku ε:

$$|x_{n+1}-x_n|\le\frac{m\cdot\varepsilon}{M}.$$

Metoda iteracije

$$x_n=\varphi(x_{n-1}),\quad n=1,2,\dots,$$

gdje je

$$|\varphi'(x)|\le q<1\ \text{ za }\ a<x<b.$$

Test za zadanu grešku ε:

$$|x_n - x_{n-1}| < \frac{1-q}{q}\varepsilon,$$

ili jednostavnije $|x_n - x_{n-1}| < \varepsilon$.

λ-trik: $f'(x) > 0$, $\lambda < \frac{2}{M_1}$, $M_1 = \max_{x\in[a,b]} |f'(x)| \Rightarrow \varphi(x) = x - \lambda f(x)$.

Sustavi nelinearnih jednadžbi

Newtonova metoda

$$X^{(k+1)} = X^{(k)} - J^{-1}(X^{(k)})F(X^{(k)}), \quad k = 0, 1, 2, \ldots,$$

gdje je

$$X^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}, \quad F(X^{(k)}) = \begin{bmatrix} f_1(X^{(k)}) \\ f_2(X^{(k)}) \\ \vdots \\ f_n(X^{(k)}) \end{bmatrix}$$

$$J(X^{(k)}) = F'(X^{(k)}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(X^{(k)}) & \frac{\partial f_1}{\partial x_2}(X^{(k)}) & \cdots & \frac{\partial f_1}{\partial x_n}(X^{(k)}) \\ \frac{\partial f_2}{\partial x_1}(X^{(k)}) & \frac{\partial f_2}{\partial x_2}(X^{(k)}) & \cdots & \frac{\partial f_2}{\partial x_n}(X^{(k)}) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(X^{(k)}) & \frac{\partial f_n}{\partial x_2}(X^{(k)}) & \cdots & \frac{\partial f_n}{\partial x_n}(X^{(k)}) \end{bmatrix}$$

Metoda iteracije

$$\begin{aligned} x_1 &= \varphi_1(x_1, x_2, \ldots, x_n) \\ x_2 &= \varphi_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n &= \varphi_n(x_1, x_2, \ldots, x_n) \end{aligned}$$