# NUMERIČKE METODE I PROGRAMIRANJE
# (formule za drugi parcijalni ispit)

## Interpolacija

### Lagrangeov oblik interpolacijskog polinoma

$$L_n(x) = \sum_{k=0}^{n} f(x_k) \prod_{j=0,\ j\neq k}^{n} \frac{x - x_j}{x_k - x_j}.$$

Lokalna ograda:

$$|f(x) - L_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |(x - x_0)(x - x_1)\cdots(x - x_n)|, \quad M_{n+1} = \max_{x\in[x_{0n}, x_n]} |f^{(n+1)}(x)|$$

Uniformna ograda (ekvidistantni čvorovi):

$$\max_{x\in[x_0,x_n]} |f(x) - L_n(x)| \leq \frac{h^{n+1}}{4(n+1)} M_{n+1} = \varepsilon_n, \quad h = x_{i+1} - x_i = \frac{x_n - x_0}{n}.$$

### Aitkenova interpolacijska shema

$$L_{i,i+1}(x) = \frac{1}{x_{i+1} - x_i} \begin{vmatrix} y_i & x_i - x \\ y_{i+1} & x_{i+1} - x \end{vmatrix}, \quad L_{i,i+1,i+2}(x) = \frac{1}{x_{i+2} - x_i} \begin{vmatrix} L_{i,i+1}(x) & x_i - x \\ L_{i+1,i+2}(x) & x_{i+2} - x \end{vmatrix}, \ldots$$

$$L_n(x) = L_{0,1,\ldots,n}(x) = \frac{1}{x_n - x_0} \begin{vmatrix} L_{0,1,\ldots,n-1}(x) & x_0 - x \\ L_{1,2,\ldots,n}(x) & x_n - x \end{vmatrix}.$$

### Opći Newtonov interpolacijski polinom

$$f[x_i] = f(x_i), \ - \text{ podijeljena razlika nultog reda,}$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \ - \text{ podijeljena razlika prvog reda,}$$

$$\vdots$$

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i} - \text{podijeljena razlika } k - \text{tog reda.}$$

Tabeliranje:

| $x_i$ | $y_i$ | $f[x_i, x_{i+1}]$ | $f[x_i, x_{i+1}, x_{i+2}]$ | $\ldots$ | |
|---|---|---|---|---|---|
| $x_0$ | $y_0$ | | | | |
| | | $f[x_0, x_1]$ | | | |
| $x_1$ | $y_1$ | | $f[x_0, x_1, x_2]$ | | |
| | | $f[x_1, x_2]$ | | $\ldots$ | |
| $x_2$ | $y_2$ | | $f[x_1, x_2, x_3]$ | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $f[x_0, x_1, \ldots, x_n]$ |
| $x_{n-2}$ | $y_{n-2}$ | | $f[x_{n-3}, x_{n-2}, x_{n-1}]$ | | |
| | | $f[x_{n-2}, x_{n-1}]$ | | $\ldots$ | |
| $x_{n-1}$ | $y_{n-1}$ | | $f[x_{n-2}, x_{n-1}, x_n]$ | | |
| | | $f[x_{n-1}, x_n]$ | | | |
| $x_n$ | $y_n$ | | | | |

$$L_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

ili

$$L_n(x) = f[x_n] + f[x_{n-1}, x_n](x - x_n) + f[x_{n-2}, x_{n-1}, x_n](x - x_n)$$
$$(x - x_{n-1}) + \cdots + f[x_0, x_1, \ldots, x_n](x - x_n)\cdots(x - x_1).$$

## Hermiteov interpolacijski polinom

$$f[x_i, x_i] := f'(x_i), \quad f[x_i, x_i, x_i] = f''(x_i)/2!, \ldots, f[x_i, \ldots, x_i] := f^{(n)}(x_i)/n!$$

Lokalna greška:

$$f(x) - L_m(x) = \frac{f^{(m+1)}(\mu)}{(m+1)!} |(x - x_0)^{m_0}(x - x_1)^{m_1} \cdots (x - x_n)^{m_n}|, \quad \mu \in (x_0, x_n)$$

gdje je $m_0 + m_1 + \cdots + m_n - 1 = m$, pri čemu su u čvorovima $x_i$ dane derivacije do uključivo $(m_i - 1)-$vog reda, $i = 0, 1, \ldots, n$.

## Interpolacija po dijelovima polinomima

$$\begin{aligned}
g|_{[x_k, x_{k+1}]} &= P_k, \quad k = 0, 1, \ldots, n-1, \\
g(x_k) &= y_k, \quad k = 0, \ldots, n, \\
P_k(x_k) = y_k, \quad P_k(x_{k+1}) &= y_{k+1}, \quad k = 0, \ldots, n-1, \\
P_{k-1}(x_k) &= P_k(x_k), \quad k = 1, \ldots, n-1.
\end{aligned}$$

### Po dijelovima kubna interpolacija

$$\begin{aligned}
P_k(x_k) = y_k, \ P_k(x_{k+1}) &= y_{k+1}, \\
P_k'(x_k) = s_k, \ P_k'(x_{k+1}) &= s_{k+1}. \\
P_{k-1}'(x_k) = P_k'(x_k), &\quad k = 1, \ldots, n-1.
\end{aligned}$$

$$P_k(x) = y_k + (x - x_k) f[x_k, x_k] + (x - x_k)^2 f[x_k, x_k, x_{k+1}] + (x - x_k)^2 (x - x_{k+1}) f[x_k, x_k, x_{k+1}, x_{k+1}].$$

Oblik

$$P_k(x) = c_{1,k} + c_{2,k}(x - x_k) + c_{3,k}(x - x_k)^2 + c_{4,k}(x - x_k)^3.$$

$$c_{1,k} = P_k(x_k) = y_k, \quad c_{2,k} = P_k'(x_k) = s_k,$$

$$c_{4,k} = \frac{s_k + s_{k+1} - 2f[x_k, x_{k+1}]}{(x_{k+1} - x_k)^2}, \quad c_{3,k} = \frac{f[x_k, x_{k+1}] - s_k}{x_{k+1} - x_k} - (x_{k+1} - x_k) c_{4,k}.$$

### Kubni splajn

Dodatni uvjet:

$$P_{k-1}''(x_k) = P_k''(x_k), \quad k = 1, \ldots, n-1.$$

$$\Delta x_k s_{k-1} + 2(\Delta x_{k-1} + \Delta x_k) s_k + \Delta x_{k-1} s_{k+1} = 3(\Delta x_k f[x_{k-1}, x_k] + \Delta x_{k-1} f[x_k, x_{k+1}]), \quad k = 1, \ldots, n-1.$$

Zadani ugibi (druga derivacija) u rubovima $f''(x_0)$, $f''(x_n)$:

$$2s_0 + s_1 = 3f[x_0, x_1] - \frac{1}{2} f''(x_0) \Delta x_0,$$

$$s_{n-1} + 2s_n = 3f[x_{n-1}, x_n] + \frac{1}{2} f''(x_n) \Delta x_{n-1}.$$

Ako sustave u kojima je matrica sustava trodijagonalna napišemo u obliku $AT = D$, gdje je

$$A = \begin{bmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1} & B_{n-1} & C_{n-1} \\ & & & A_n & B_n \end{bmatrix}, \quad T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{n-1} \\ T_n \end{bmatrix}, \quad D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix},$$

rekurzivna metoda rješavanja je:

$$b_1 = \frac{C_1}{B_1}, \quad b_i = \frac{C_i}{B_i - A_i b_{i-1}}, \quad i = 2, 3, \ldots, n-1,$$

$$q_1 = \frac{D_1}{B_1}, \quad q_i = \frac{D_i - A_i q_{i-1}}{B_i - A_i b_{i-1}}, \quad i = 2, 3, \ldots, n,$$

$$T_n = q_n, \quad T_i = q_i - b_i T_{i+1}, \quad i = n-1, \ldots, 1.$$

# Numeričko diferenciranje

Tri točke:

$$f'(x_0) = \frac{1}{2h}\left(-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right) + \frac{h^2}{3} f^{(3)}(\xi(x_0))$$

$$f'(x_0) = \frac{1}{2h}\left(f(x_0+h) - f(x_0-h)\right) - \frac{h^2}{6} f^{(3)}(\xi(x_1))$$

Četiri točke:

$$f'(x_0) = \frac{1}{12h}\left(f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h) + \frac{h^2}{30} f^{(4)}(\xi)\right.$$

# Simpsonova formula

$$I_{2n} = \frac{b-a}{6n}\left[f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 4\sum_{i=0}^{n-1} f(x_{2i+1})\right].$$

Greška:

$$\left|\int_a^b f(x)dx - I_{2n}\right| \le \frac{b-a}{180} M_4 h^4,$$

pri čemu je $M_4 = \max\{|f^{(4)}(x)| : x \in [a,b]\}, \ h = \frac{b-a}{2n}$.

# Analitičke metode za približno rješavanje običnih diferencijalnih jednadžbi

$$y' = f(x,y), \quad y(x_0) = y_0.$$

## Taylorova metoda

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{1}{2!}(x-x_0)^2 y''(x_0) + \cdots$$

$$\begin{aligned} y(x_0) &= y_0, \\ y'(x_0) &= f(x_0, y_0), \\ y''(x_0) &= \left[\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y}\right]_{x=x_0, y=y_0} = \frac{\partial f(x_0,y_0)}{\partial x} + f(x_0,y_0)\frac{\partial f(x_0,y_0)}{\partial y}, \quad \text{itd} \end{aligned}$$

## Metoda neodređenih koeficijenata

Rješenje problema tražimo u obliku

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots,$$

gdje nepoznate koeficijente $a_k \ \ (k = 0, 1, \ldots)$ određujemo iz jednadžbe i zadanog početnog uvjeta.

**Picardova metoda (metoda sukcesivne aproksimacije)**

$$y_n(x) = y_0 + \int_{x_0}^{x} f[x, y_{n-1}(x)]dx, \quad n = 1, 2, \ldots$$

## Laplaceova transformacija

$$F(p) = \int_0^{+\infty} e^{-pt} f(t)dt$$

Teorem o diferenciranju transformata: Neka je $[\mathcal{L}(f)](p) = F(p)$. Tada vrijedi

$$[\mathcal{L}(t^n f(t))](p) = (-1)^n F^{(n)}(p).$$

Teorem o integriranju transformata: Neka je $[\mathcal{L}(f)](p) = F(p)$. Tada vrijedi

$$\left[\mathcal{L}\left(\frac{f(t)}{t}\right)\right](p) = \int_p^{+\infty} F(p)dp.$$

Teorem o homotetiji (sličnosti): Neka je $a \in \mathbf{R}^+$ i neka je $[\mathcal{L}(f)](p) = F(p)$. Tada vrijedi

$$\mathcal{L}\left(f(at)\right)(p) = \frac{1}{a} F\left(\frac{p}{a}\right).$$

Teorem o pomaku: Neka je $[\mathcal{L}(f)](p) = F(p)$ i neka je funkcija $f$ definirana za $p > s_0$. Tada vrijedi

$$\left[\mathcal{L}\left(e^{at} f(t)\right)\right](p) = F\left(p - a\right), \quad \forall p > s_0 + a.$$

Teorem o prigušenju: Ako je $a > 0$ i ako je $[\mathcal{L}(f)](p) = F(p)$, tada vrijedi

$$\left[\mathcal{L}\left(e^{-at} f(t)\right)\right](p) = F\left(p + a\right).$$

Teorem o diferenciranju originala: Neka je funkcija $f : [0, +\infty) \to \mathbf{R}$ eksponencijalnog rasta reda $s_0$ i neka na $(0, +\infty)$ ima neprekidnu prvu derivaciju. Za $p > s_0$ Laplaceov transformat od $f'$ postoji i on je

$$\mathcal{L}\left(f'(t)\right) = p\mathcal{L}(f) - f(0), \quad \forall p > s_0.$$

Matematičkom se indukcijom lako pokazuje da je

$$\mathcal{L}\left(f^{(n)}(t)\right) = p^n \mathcal{L}(f) - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - f^{(n-1)}(0), \quad \forall p > s_0.$$

Teorem o integriranju originala: Neka je $f : [0, +\infty) \to \mathbf{R}$ lokalno integrabilna funkcija eksponencijalnog rasta reda $s_0$. Tada funkcija

$$t \to g(t) = \int_0^t f(\tau)d\tau$$

ima Laplaceov transformat i vrijedi

$$\left[\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)\right](p) = \frac{1}{p}[\mathcal{L}(f)](p), \quad \forall p > s_0 > 0.$$

Teorem o produktu: Neka su $[\mathcal{L}(f_1)](p) = F_1(p)$ i $[\mathcal{L}(f_2)](p) = F_2(p)$. Tada je

$$[\mathcal{L}(f)](p) = F_1(p)F_2(p),$$

gdje je $f(t) = \int_0^t f_1(u) f_2(t-u)du$.

**Tablica $\mathcal{L}$-transformacije**

| $f(t)$ | $F(p) = [\mathcal{L}(f)](p)$ |
|---|---|
| $a$ | $\frac{a}{p}$ |
| $e^{at}$ | $\frac{1}{p-a}$ |
| $t^n,\ n \in \mathbf{N}$ | $\frac{n!}{p^{n+1}}$ |
| $\sin wt$ | $\frac{w}{p^2+w^2}$ |
| $\cos wt$ | $\frac{p}{p^2+w^2}$ |
| $e^{\pm at}\sin wt$ | $\frac{w}{(p\mp a)^2+w^2}$ |
| $e^{\pm at}\cos wt$ | $\frac{p\mp a}{(p\mp a)^2+w^2}$ |

## Rješavanje linearnih diferencijalnih jednadžbi s konstantnim koeficijentima

$$\begin{aligned}
\mathcal{L}(x') &= pX - x_0 \\
\mathcal{L}(x'') &= p^2X - px_0 - x_0' \\
\cdots\cdots & \quad \cdots\cdots \\
\mathcal{L}(x^{(n-1)}) &= p^{n-1}X - p^{n-2}x_0 - \cdots - x_0^{(n-2)} \\
\mathcal{L}(x^{(n)}) &= p^nX - p^{n-1}x_0 - \cdots - x_0^{(n-1)}
\end{aligned}$$

# Numeričke metode za obične diferencijalne jednadžbe

$$y' = f(x, y),\ y(x_0) = y_0.$$

## Eulerova metoda

$$y_{i+1} = y_i + hf(x_i, y_i), \quad i = 1, \ldots, n,$$

## Runge-Kuttine metode

$$y_{i+1} = y_i + \Delta y_i.$$

Heunova metoda:

$$\begin{aligned}
\Delta y_i &= h\Phi = \frac{1}{2}(K_1^{(i)} + K_2^{(i)}), \\
K_1^{(i)} &= hf(x_i, y_i), \\
K_2^{(i)} &= f(x_i + h, y_i + K_1^{(i)}),
\end{aligned}$$

Modificirana Eulerova metoda:

$$\Delta y_i = h\Phi = hf\left(x_i + \frac{h}{2}, y_i + \frac{hf(x_i, y_i)}{2}\right).$$

Runge-Kutta ili RK-4 metoda:

$$\begin{aligned}
\Delta y_i = h\Phi &= \frac{1}{6}(K_1^{(i)} + 2K_2^{(i)} + 2K_3^{(i)} + K_4^{(i)}), \\
K_1^{(i)} &= hf(x_i, y_i), \\
K_2^{(i)} &= hf\left(x_i + \frac{h}{2}, y_i + \frac{K_1^{(i)}}{2}\right), \\
K_3^{(i)} &= hf\left(x_i + \frac{h}{2}, y_i + \frac{K_2^{(i)}}{2}\right), \\
K_4^{(i)} &= hf(x_i + h, y_i + K_3^{(i)}).
\end{aligned}$$

## Shema konačnih razlika

$$\frac{\partial u(x_i, y_j)}{\partial x} \approx \frac{u_{i+1j} - u_{ij}}{h}, \frac{\partial u(x_i, y_j)}{\partial y} \approx \frac{u_{ij+1} - u_{ij}}{k}$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial x^2} \approx \frac{u_{i-1j} - 2u_{ij} + u_{i+1j}}{h^2}, \quad \frac{\partial^2 u(x_i, y_j)}{\partial y^2} \approx \frac{u_{ij-1} - 2u_{ij} + u_{ij+1}}{k^2},$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial x \partial y} \approx \frac{u_{i+1j+1} - u_{i-1j+1} - u_{i+1j-1} + u_{i-1j-1}}{4hk}.$$

## Metoda zlatnog reza

Algoritam za metodu zlatnog reza:

Neka točke $a^{(0)}, b^{(0)}$ i $c^{(0)}$ zadovoljavaju

$$\frac{b^{(0)} - a^{(0)}}{c^{(0)} - a^{(0)}} = w = \frac{3 - \sqrt{5}}{2}$$

te

$$f(b^{(0)}) \leq f(a^{(0)}) \quad \text{i} \quad f(b^{(0)}) \leq f(c^{(0)}).$$

$i = 0$
DOUNTIL $|c^{(i+1)} - a^{(i+1)}| \leq \varepsilon$
$x^{(i)} = c^{(i)} + a^{(i)} - b^{(i)}$
IF $f(x^{(i)}) < f(b^{(i)})$
$a^{(i+1)} = b^{(i)}, \quad b^{(i+1)} = x^{(i)}, \quad c^{(i+1)} = c^{(i)}, \quad i = i + 1$
ELSE
$a^{(i+1)} = a^{(i)}, \quad b^{(i+1)} = a^{(i)} + x^{(i)} - b^{(i)}, \quad c^{(i+1)} = x^{(i)}, \quad i = i + 1$
ENDIF
ENDDO
$x^* = (a^{(i+1)} + c^{(i+1)})/2$
END