**Zadatak 1** *Za funkciju* $f(x) = \frac{1}{\sqrt{x+4}}$ *poznate su vrijednosti* $f(1)$ *i* $f(2)$*. Odredite* $f'(0)$*:*

*a) Hermiteovom metodom ako je još poznato i* $f'(2)$*,*

*b) koristeći kubni splajn ako su poznate vrijednosti* $f(0)$*,* $f''(0)$ *i* $f''(2)$*,*

*c) numeričkim diferenciranjem ako je poznata vrijednost* $f(0)$*.*

*Izračunajte pravu grešku u sva tri slučaja.*

*Rješenje.* a) $f'(x) = -\frac{1}{2\sqrt{(x+4)^3}}$

| $x_i$ | $y_i$ | $f^{[1]}$ | $f^{[2]}$ |
|---|---|---|---|
| $x_{-1} = 0$ | $y_{-1} =?$ | | |
| | | $f'(x_{-1}) =?$ | |
| $x_{-1} = 0$ | $y_{-1} =?$ | | $f[x_{-1}, x_{-1}, x_1] =?$ |
| | | $f[x_{-1}, x_1] =?$ | |
| $x_1 = 1$ | $y_1 = 0.44721$ | | $f[x_{-1}, x_1, x_2] =?$ |
| | | $f[x_1, x_2] = -0.03896$ | |
| $x_2 = 2$ | $y_2 = 0.40825$ | | $f[x_1, x_2, x_2] = 0.00494$ |
| | | $f'(2) = -0.03402$ | |
| $x_2 = 2$ | $y_2 = 0.40825$ | | |

$$\frac{-0.03896 - f[x_{-1}, x_1]}{2} = 0.00494 \Rightarrow f[x_{-1}, x_1] = -0.04884$$
$$\Rightarrow \frac{-0.04884 - f'(x_{-1})}{1} = 0.00494 \Rightarrow f'(0) = -0.05378.$$

Kako je prava vrijednost $f'(0) = -0.0625$ za pravu grešku imamo $|-0.05378+0.0625| = 0.872\cdot 10^{-2}$.

b)

| $x_i$ | $y_i$ | $f[x_i, x_{i+1}]$ |
|---|---|---|
| $x_0 = 0$ | $y_0 = 0.5$ | |
| | | $f[x_0, x_1] = -0.05279$ |
| $x_1 = 1$ | $y_1 = 0.44721$ | |
| | | $f[x_1, x_2] = -0.03896$ |
| $x_2 = 2$ | $y_2 = 0.40825$ | |

$$\Rightarrow s_0 + 4s_1 + s_2 = 3(-0.05279 + 0.03896) = -0.27525.$$

Kako je i $f''(x) = \frac{3}{4\sqrt{(x+4)^5}}$ imamo

$$2s_0 + s_1 = 3(-0.05279) - \frac{1}{2} \cdot 0.02344 = -0.17009,$$

$$s_1 + 2s_2 = 3(-0.03896) + \frac{1}{2} \cdot 0.0085 = -0.11263.$$

Rješenje sustava je $s_0 = -0.06273$, a prava grška $|-0.06273 + 0.0625| = 0.23 \cdot 10^{-3}$.

c)

$$f'(0) = \frac{1}{2 \cdot 1}(-3f(0) + 4f(1) - f(2)) = -0.0597.$$

Greška: $|-0.0597 + 0.0625| = 0.279 \cdot 10^{-2}$.

**Zadatak 2** *Simpsonovom metodom s točnošću većom od* $10^{-4}$ *izračunajte* $\int_0^\pi \cos\frac{x}{5}dx$*. Odredite pravu grešku.*

*Rješenje.*

$$f(x) = \cos\frac{x}{5} \Rightarrow f'(x) = -\frac{1}{5}\sin\frac{x}{5} \Rightarrow f''(x) = -\frac{1}{25}\cos\frac{x}{5} \Rightarrow f'''(x) = \frac{1}{125}\sin\frac{x}{5} \Rightarrow f^{iv}(x) = \frac{1}{625}\cos\frac{x}{5}$$

$$\Rightarrow M_4 = f(0) = \frac{1}{625} \Rightarrow \frac{h^4}{180} \cdot \frac{\pi}{625} < 10^{-4} \rightarrow\Rightarrow 2n > 2.2837 \Rightarrow 2n = 4.$$

| $x_i$ | $f(x_i)$ |
|---|---|
| $x_0 = 0$ | $f(x_0) = 1$ |
| $x_1 = \pi/4$ | $f(x_1) = 0.98769$ |
| $x_2 = \pi/2$ | $f(x_2) = 0.95106$ |
| $x_3 = 3\pi/4$ | $f(x_3) = 0.89101$ |
| $x_4 = \pi$ | $f(x_4) = 0.80902$ |

$$\Rightarrow I_4 = \frac{\pi}{12}(1+4(0.98769+0.89101)+2\cdot 0.95106+0.80902) = 2.93894.$$

Kako je $\int_0^\pi \cos\frac{x}{5}dx = 5\sin\frac{x}{5}\Big|_0^\pi = 2.93893$, prava greška je $|2.93894 - 2.93893| = 0.14\cdot 10^{-4}$.

**Zadatak 3** *Koristeći Laplaceovu transformaciju odredite rješenje diferencijalne jednadžbe $x''(t) - x(t) = -5\sin 2t$ uz početne uvjete $x(0) = 1, x'(0) = 2$.*

*Rješenje.*

$$\mathcal{L}(x'') = p^2X - px_0 - x_0' = p^2X - p - 2 \Rightarrow p^2X - p - 2 - X = -5\cdot\frac{2}{p^2+4}$$

$$\Rightarrow X = \frac{p^3 + 2p^2 + 4p - 2}{(p^2+4)(p^2-1)} = \frac{2}{p^2+4} + \frac{1}{2}\frac{1}{p-1} + \frac{1}{2}\frac{1}{p+1}$$

$$\Rightarrow x(t) = \sin 2t + \frac{e^t}{2} + \frac{e^{-t}}{2}.$$

**Zadatak 4** *Diferencijalnu jednadžbu $y' = \frac{1}{y}$, $y(0) = 1$ na intervalu $[0,1]$ s korakom $h = 0.5$ približno riješite Runge-Kutta metodom, te Picardovom metodom u dvije iteracije i ocjenite koja je metoda točnija u točki $x = 0.5$ (izračunajte pravu grešku).*

*Rješenje.* Pravo rješenje:

$$ydy = dx \Rightarrow \frac{y^2}{2} = x + C \Rightarrow C = \frac{1}{2} \Rightarrow y = \sqrt{2x+1} \Rightarrow y(0.5) = 1.41421.$$

Picardova metoda:

$$y_1(x) = 1 + \int_0^x dx = 1 + x \Rightarrow y_2(x) = 1 + \int_0^x \frac{1}{1+x}dx = 1 + \ln(1+x) \Rightarrow y_2(0.5) = 1.40546,\ y_2(1) = 1.69315.$$

Prava greška: $|1.40546 - 1.41421| = 0.875\cdot 10^{-2}$.

Runge-Kuttina metoda:

$$K_1^0 = 0.5,\quad K_2^0 = 0.4,\quad K_3^0 = 0.41667,\quad K_4^0 = 0.35294$$

$$\triangle y_0 = 0.41438 \Rightarrow y_1 = 1.41438$$

$$K_1^1 = 0.35351,\quad K_2^1 = 0.31424,\quad K_3^1 = 0.31817,\quad K_4^1 = 0.28859$$

$$\triangle y_1 = 0.31782 \Rightarrow y_2 = 1.7322$$

Prava greška: $|1.41438 - 1.41421| = 0.17\cdot 10^{-3}$.

Točnija je Runge-Kuttina metoda.

**Zadatak 5** *Metodom zlatnog reza s greškom manjom od $\varepsilon = 0.5$ odredite minimum funkcije $f(x) = x^2 + \frac{2}{x+1}$ na intervalu $[0,1]$.*

*Rješenje.* Kako je $a^{(0)} = 0$ i $c^{(0)} = 1$ imamo

$$\frac{b^{(0)} - 0}{1} = \frac{3-\sqrt{5}}{2} \Rightarrow b^{(0)} = 0.38197.$$

Kako je još

$$f(a^{(0)}) = f(0) = 2,\quad f(c^{(0)}) = f(1) = 2,\quad f(b^{(0)}) = 1.59311$$

$$\Rightarrow f(b^{(0)}) < f(a^{(0)}) \text{ i } f(b^{(0)}) < f(c^{(0)}),$$

početne su točke dobro odabrane.

Sada,

$$x^{(0)} = c^{(0)} + a^{(0)} - b^{(0)} = 0.61803,\quad f(x^{(0)}) = 1.61803 > f(b^{(0)})$$

$$\Rightarrow a^{(1)} = 0,\quad b^{(1)} = a^{(0)} + x^{(0)} - b^{(0)} = 0.23606,\quad c^{(1)} = 0.61803,\quad |c^{(1)} - a^{(1)}| = 0.61803 > 0.5$$

$$\Rightarrow x^{(1)} = c^{(1)} + a^{(1)} - b^{(1)} = 0.38197,\quad f(x^{(1)}) = 1.59311 < f(b^{(1)}) = 1.67377$$

$$\Rightarrow a^{(2)} = 0.23607,\quad b^{(2)} = 0.38197,\quad c^{(2)} = 0.61803,\quad |c^{(2)} - a^{(2)}| = 0.38197 < 0.5$$

$$\Rightarrow x^* = (a^{(2)} + c^{(2)})/2 = 0.42705.$$